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MODEL OF SECURITY VALUATION

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A TIME-STATE-PREFERENCE MODEL OF SECURITY VALUATION

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A TIME-STATE-PREFERENCE MODEL OF SECURITY VALUATION[1]

By Stewart C. Myers[2]

I. INTRODUCTION AND SUMMARY

Determining the value of streams of returns yielded by financial assets is a task common to many sorts of economic analysis. The literature on this subject is extensive at all levels of abstraction. Most of this work, however, has not come to grips with one pervasive consideration--that is, uncertainty.

This paper presents a model of security valuation in which uncertainty takes the central role. The model is based on the requirements for security market equilibrium in a "time-state-preference" framework, in which uncertainty is described by a set of event-sequences, or states of nature. The framework used here is a generalized version of that used in articles by Arrow, Debreu and Hirshleifer, as well as in several more recent studies.[3]

The valuation formulas presented here are, of course, imperfect. They cannot be represented as handy empirical tools. On the theoretical front, moreover, new results and new problems seem always to arrive hand in hand. Although the problems are duly noted, the time-state-preference model will be defended as a plausible approximation and a useful analytical tool.

The paper is organized as follows. In Section II, existing valuation models are briefly reviewed. The basic time-state-preference model is

derived and discussed in Section III. This requires careful statement of the assumed market characteristics and the constraints on investors' strategies: although the general characteristics of the formulas obtained are intuitively appealing, their precise form is sensitive to the range of trading opportunities open to investors. The effects of constraints on these opportunities are considered. In Section IV, the special case discussed by Arrow, Debreu and Hirshleifer[4] is related to the more general framework presented here. Some general implications of the model are considered in Section V.

Finally, we consider the possible effects of what I call "the interdependence of investors' strategies," which arises whenever the value of a security to an investor depends on other investors' beliefs and market strategies. This interdependence leads to price uncertainty, which is difficult to incorporate in the traditional, static models of equilibrium. Thus, it is difficult to evaluate its systematic effect, if any, on the structure of security prices. It is possible, however, to make qualitative comments on the nature of the problem and its possible effects.

The main contributions of this paper, as I see them, are as follows:

1. The paper is a general investigation of how markets for risky assets work. It is more exploratory than definitive, which should not be surprising: work on this topic has a relatively short history, and has concentrated mostly on conceptual issues. However, this paper is one of the first detailed investigations of a particular market under relatively plausible conditions.
2. It is widely agreed that the time-state-preference framework developed by Arrow, Debreu and Hirshleifer[5] is an important

addition to the economist's theoretical tool-kit. This paper shows that the framework is amenable to considerable generalization and adaptation to particular market conditions.

3. The model presented below was originally developed as a contribution to the theory of corporate financial management. Although details are not included here, it has proved useful in this context.[6] Thus it should be worthwhile to set out the logic of the model in detail as a basis for further work.

II. OTHER APPROACHES

Normally, the present value of a security is defined as:

$$P = \sum_{t=1}^{\infty} \frac{\bar{R}(t)}{(1+r)^t} \quad (1)$$

where $\bar{R}(t)$ represents the expected return at period t, and r is a "required rate of return," assumed to include a risk premium to compensate the investor for bearing the uncertainty inherent in the stream of returns. For stocks, this usually is taken to mean that price equals the discounted value of future dividends.[7]

This approach is undoubtedly useful. The trouble is that writers who use it normally offer only the most casual explanations of what determines whether r is small or large. It does not help to say that "k is a positive function of the riskiness of the stream of expected returns" unless "riskiness" can be defined--and this is not exactly simple.[8]

Another variant of discounted present value uses certainty equivalents

of expected returns. Here,

$$P = \sum_{t=1}^{\infty} \frac{\alpha(t)\bar{R}(t)}{(1+i)^t} \quad (2)$$

with i defined as the riskless rate of interest, and the terms $\alpha(t)\bar{R}(t)$ as certainty equivalents of the expected returns $\bar{R}(t)$. The coefficients $\alpha(t)$ are chosen so that the investor is indifferent to the expected returns $\bar{R}(t)$ and certain returns amounting to $\alpha(t)\bar{R}(t)$ for each future period t.

Although the use of certainty equivalents has advantages,[9] Eq. (2) is not of much help unless the determinants of the coefficients $[\alpha(1), \ldots, \alpha(t), \ldots]$ are known.

In most discussions, the formula for the present value of a stream of returns is rigorously derived only for conditions of certainty.[10] Addition of the coefficients $\alpha(t)$ to the numerator, or a risk premium to the discount rate, is simply a heuristic, though plausible, adaptation of the formula for the certainty case. The model presented in the next section puts present value under uncertainty on a more rigorous basis.

Another approach to valuation has been taken by Sharpe and Lintner, who have used the procedures for portfolio selection originally developed by Markowitz as a framework for formal models of capital markets under uncertainty.[11] Briefly, the characteristics of these models are as follows:

1. Each investor is assumed to purchase securities on the basis of their anticipated returns in one (presumably short) period from t = 0 to t = 1, and not on the basis

of anticipated returns in any subsequent periods. In other words, the investor expects to choose a new portfolio at t = 1 (but not before), and is indifferent to the amount of trading which may be required to obtain the portfolio which will be optimal for him at that time.

2. The value of any feasible portfolio to an investor is a positive function of the expected percentage return (dividends or interest, plus or minus changes in market value) of the portfolio, and a negative function of the variance of this return. Other possible determinants of the value of portfolios to investors are not considered.
3. The value of the k^{th} security depends on the expected return of the security, the variance of this return and the covariances of the k^{th} security's return with returns on the other securities which may be purchased.

This sort of model is an extremely valuable tool for analyzing the mechanisms by which investors' demands for portfolios of risky securities affect the market prices of these securities. In many instances, however, the potential usefulness of these models is limited by the fact that they explain prices only in terms of investors' expectations of future prices. From the point of view of a corporation's financial manager, for instance, portfolio analysis models are directly useful only if investors' present expectations of future prices can be explained; and this is just as formidable a task as explaining present prices.

Lintner has recognized this problem, proposing that a security price at any future time be regarded as a simple function of the mean, variance,

and covariances of subsequent returns. By an iterative argument, future prices may be eliminated from the valuation formula.[12] This extension of the portfolio analysis framework is an alternative to the model described below. In my opinion, the time-state-preference model is a more flexible and powerful tool. However, there are surely sufficient problems to support both approaches.

III. THE BASIC TIME-STATE-PREFERENCE MODEL

One way of describing uncertainty about conditions in a future period is to say that one of a set of possible states of nature will occur at that time. Definition of a set of states of nature in turn provides a means of describing the risk characteristics of securities, since any security can be regarded as a contract to pay an amount which depends on the state which actually occurs.

For instance, we might regard a share of stock as a contract to pay an x dollar dividend if state 1 occurs at t = 1, a y dollar dividend if state 2 occurs at t = 1, etc. Let the dividend paid be R(s,t) and suppose 100 states of nature are being considered for t = 1. Then the set $\{R(s,1)\} = \{R(1,1), R(2,1), \ldots, R(100,1)\}$ specifies the particular bundle of <u>contingent payments</u> which the investor obtains for t = 1 by purchasing one share. In this case, R(1,1) = x, R(2,1) = y, and so on.

The following model relates the present value of a security to the present value of the contingent returns the security may pay to its owner. This relationship will be derived from the <u>necessary</u> conditions of security market equilibrium. First, however, the assumed characteristics of the market must be carefully specified.

Assumptions

1. States of Nature. -- A state of nature which may occur at time τ is defined as a particular sequence of events during the time span from t = 1 to t = τ. Constructing a set of possible states is simply a means of identifying the possible event-sequences relevant to present decisions.

The concept of an event-sequence is ambiguous, however, if "event" is left undefined, since a possibility that is relevant in one context may not be in another. The consequences of some decisions depend only on whether a stock's price rises or falls. In other cases, the decision-maker may consider the possible prices of a score of securities and the possible fortunes of a family business as well.

A benchmark can be established by imagining a set of states defined in such great detail that the knowledge of which state will occur at any time t would allow the specification of every characteristic of the future world from the present to time t. Let this set be S. The sets of states which would be considered relevant to actual decisions may be regarded as partitions of S. Thus if an investor finds it useful to identify a state by "GM's dividend is increased at t = 1," the state refers to that subset of S for which this "event" takes place.

In the model presented here, it is assumed that investors agree on a particular partition,[13] which defines a set of states $\{(s,t)\}$. The set is assumed to apply to the time span from t = 1 to t = T.[14] Conditions at t = 0 are known with certainty. The set $\{(s,t)\}$ is sufficiently detailed that, if state s occurs at time t, then returns on every security are uniquely specified for period t and all previous periods. Also,

the set of states is finite and exhaustive with respect to possible sequences of security returns.

Given these conditions, a security's contingent returns $\{R(s,t)\}$ are not random variables; the return $R(s,t)$ is certain to be paid in period t if state s occurs. However, it is important to remember that the set $\{(s,t)\}$ does not catalogue all possible future events. Even if it could be known that a particular state (s,t) is to occur, an investor would still face a residual uncertainty about his health, tastes, family status, employment, etc.

2. The Economy. -- We will imagine an economy split into real and financial sectors. For present purposes, "financial sector" and "security markets" are synonymous.

It is clearly meaningless to speak of the equilibrium of security markets except in relation to a particular set of conditions in the real sector. Accordingly, the following items are taken as given:

a. The set of states $\{(s,t)\}$.

b. Investors' assessments of the probabilities that the various states will occur.

c. The (sequences of) security returns contingent on each state (s,t).

Also, it is assumed that investors have given endowments of wealth available for allocation among securities and other uses, which will be referred to collectively as "consumption."

3. Available Securities. -- Taking conditions in the economy's real sector as given necessitates a restriction on the types of securities that

may be issued in response to security prices at t = 0. There is no need to hold supplies of all securities constant; however, it is not consistent with the framework used here to admit changes in the supply of securities that are part and parcel of changes in the allocation of resources within the real sector.

To illustrate, suppose that interest rates fall at t = 0. In response, a firm issues bonds to finance purchases of additional plant and equipment. These additional real assets enable the firm to pay higher returns to its security-holders in some or all future contingencies, so that a link is created between current interest rates and the bundle of contingent returns which the firm offers to present investors. This is unacceptable if the analysis is to be limited to conditions for equilibrium in the financial sectors.

If, on the other hand, the firm uses the bond issue to retire a portion of its outstanding common stock, conditions in the real sector may be considered unchanged. The substitution of debt for equity in a firm's capital structure is a financing decision, and changes in the firm's real assets or investment strategy are not a necessary consequence.[15]

To summarize, changes in the supply of securities, or the issue of new types of securities, are not ruled out in what follows. It is assumed, however, that such adjustments are not of the sort that imply changed conditions in the real sector. It has already been noted that the concept "equilibrium of security markets" is meaningful only if conditions in the real sector are given.

4. Market Characteristics. -- Markets are assumed to be perfect.

5. Reinvestment of Contingent Returns. -- Investment in securities amounts to the purchase of bundles of contingent returns, which may, in general, be either consumed or reinvested when and if they are realized. For this model, however, we will effectively rule out reinvestment by assuming that investors hold their original portfolios unchanged until t = T. (This assumption is reconsidered in Section VI below.) Accordingly, a security's return in (s,t) will be interpreted as the cash payment (i.e., dividend, interest or principal payment) which its owner receives in (s,t). Capital gains or losses will not be considered, except that the price of the security in the most distant future time period under consideration will be treated as if it were a liquidating dividend.[16]

For stocks, this assures that market value is determined solely by the present value of future dividends.

6. Utility Functions. -- Investors choose portfolios which we assume maximize the expected utility of future returns on the portfolio. In addition, the total expected utility associated with any portfolio is a linear function of utility functions defined for each state. Specifically, if $\pi(s,t)$ is an investor's judgment of the probability of occurrence of contingency (s,t), and U(s,t) is the utility of returns to be received in (s,t), then the overall utility of a portfolio's contingent returns will be given by

$$\sum_{s,t} \pi(s,t)\ U(s,t).$$

The notation $\sum_{s,t}$ denotes summation over all states in the set $\{(s,t)\}$, t = 1, 2, . . . T.

Further, we assume that each utility function U(s,t) is defined only

in terms of returns to be received in (s,t). That is, if an investor holds a portfolio yielding y in (s,t), then the utility of y will be independent of possible outcomes in other contingencies. This assumption would not be reasonable without our proviso that contingent returns on securities are consumed, rather than reinvested. If, say, part of the amount y were invested in real assets, the investor's income in subsequent contingencies would be increased. As a consequence, the marginal utility of income in these contingencies would not be the same, in general, as if these funds were consumed--i.e., invested in goods of no lasting value.

Note that, in this framework, $U'(s,t)$, the marginal utility of income in a given contingency, may be high for either or both of two reasons:[17]

a. Assuming that the investor is risk-averse, $U'(s,t)$ will be relatively high to the extent that the total income to be received in (s,t) is low.

b. The utility of a given amount of money income may differ from state to state, since the utility <u>functions</u> $U(s,t)$ are not necessarily the same for each contingency.

One class of reasons why the functions $U(s,t)$ may depend on (s,t) is fairly obvious: differences can arise, for instance, if commodity prices differ from state to state and over time, or if the investor's need for income depends on, say, his age at (s,t).

Another kind of reason follows from the way we have set up the problem. The set $\{(s,t)\}$ assumed for purposes of analysis is exhaustive in the sense that it offers a complete catalogue of possible future returns on <u>securities</u>, but it does not catalogue <u>all</u> future events exhaustively. The risks inherent in these "uncatalogued" contingencies will not, in gen-

eral, be independent of the state (s,t) being considered. An investor will perhaps be less certain of the amount of income he will receive from sources other than securities in wartime, but the occurrence of a war will also effect returns on securities. We would expect the functions U(s,t) to reflect such interrelationships.

Formally, then, the phrase "utility of a contingent return A in (s,t)" must be taken to mean "the expected utility to the investor of the (certain) amount A at time t given that state s occurs." We thus consider only a part of the investor's over-all decision problem: the possible incremental effects on his future income of his portfolio choice at t = 0.

The Basic Model

We begin by considering N different securities which investors can purchase at t = 0. These securities may have been issued at t = 0, or they may be "left over" from previous periods. The word "share" will be used to refer to one unit of investment in a given security, although these securities will, of course, include bonds, money and time deposits, for which "share" is not the appropriate word in common usage.

For the k^{th} security, the set $\{R_k(s,t)\}$ of contingent returns per share will be written in vector form, and referred to as R_k, where

$$R_k = [R_k(0), \ldots, R_k(s,t), \ldots]$$

for s = 1, 2, . . ., m(t), and for each period t = 1, 2, . . ., T. The "state" s = 0 refers to the present -- i.e., to t = 0 -- and for each security $R_k(0) = -P_k$, where P_k is the ex-dividend market price per share of the k^{th} security at t = 0.

We define a dummy security k = 0 to be "consumption" at t = 0, with

$$R_0 = [1, 0, \ldots, 0].$$

That is, purchasing one share of security zero is interpreted as the consumption of one dollar at t = 0. The "price" of consumption is likewise one dollar, so $P_0 = 1$.

Consider the portfolio selection problem of a particular investor. Let h_k be the number of shares of the k^{th} security which he purchases. His decision problem at t = 0 is to choose $[h_0, h_1, \ldots, h_N]$ to maximize expected utility Ψ, where

$$\Psi = \sum_{s,t} \pi(s,t)\, U(s,t) + U(0), \tag{3}$$

with $U(s,t) = f(\sum_{k=1}^{N} h_k R_k(s,t))$ and $U(0) = f(h_0)$. The variables $\pi(s,t)$ represent the investor's assessments of the probabilities that the states (s,t) actually will occur.

In addition, the investor is constrained in that he has only a given amount of wealth, W, available for allocation among consumption and investment. The constraint is

$$\emptyset = \sum_{k=0}^{N} h_k P_k - W = 0. \tag{4}$$

Since consumption and investment in securities are the only available uses for this wealth, Eq. (4) is necessarily an equality.

If no short selling or borrowing is permitted, then $h_k \geq 0$ for all k. In this case, maximizing Eq. (3) subject to the stated constraint is a problem in non-linear programming, and the necessary conditions for the

maximum may be inferred from the Kuhn-Tucker conditions.[18] If a maximum exists, we know from these conditions that we can assign a positive number $\lambda(\emptyset)$ to the constraint Eq. (4).[19] Maximizing utility implies that

$$\frac{\delta \Psi}{\delta h_k} - \lambda(\emptyset) \frac{\delta \emptyset}{\delta h_k} \leq 0, \tag{5}$$

for $k = 0, 1, \ldots, N$. The left hand side of Eq. (5) is zero if $h_k > 0$.

Note that $\frac{\delta \Psi}{\delta h_k} = P_k$ for $k > 0$, and $\frac{\delta \emptyset}{\delta h_k} = 1$. Substituting in Eq. (5) for security $k = 0$ (i.e., consumption at $t = 0$), we obtain

$$\lambda(\emptyset) = U'(0), \tag{6}$$

where $U'(0)$ is the marginal utility of income used for present consumption.

Using these results, we can rewrite Eq. (5) as

$$\delta \Psi / \delta h_k - U'(0) P_k \leq 0,$$

or

$$P_k \geq 1/U'(0)[\delta \Psi / \delta h_k].$$

Since, for $k \neq 0$, $\delta \Psi / \delta h_k = \sum_{s,t} \pi(s,t) U'(s,t) R_k(s,t)$, we have the fundamental result

$$P_k \geq \sum_{s,t} q(s,t) R_k(s,t), \tag{7}$$

where

$$q(s,t) = \pi(s,t) \frac{U'(s,t)}{U'(0)}. \tag{8}$$

Equation (7) is the basic valuation formula for the time-state-

preference framework. In words, it tells us that when an investor maximizes the expected utility of his portfolio, the price of each security is at least equal to the expectation of the marginal utility associated with a small increment in his holdings of that security, where the utility of money in future contingencies is measured in terms of the utility of money used for present consumption. If the investor actually holds that security in his portfolio, then its price is exactly equal to the expectation of the marginal utility associated with the security. The terms $q(s,t)$ thus indicate the present value to this investor of an incremental dollar of portfolio return to be received at time t if state (s,t) occurs.

A necessary condition for equilibrium is that Eq. (7) holds for all securities from the point of view of each investor. In effect Eq. (7) establishes a lower bound on the price of each security, expressed in terms of investors' marginal valuations of contingent returns. For if P_k were less than the right hand side of Eq. (7) from the point of view of any investor, then that investor could increase the total expected utility of returns to his portfolio by purchasing security k in at least marginal amounts. Equilibrium cannot exist until all such opportunities are exhausted.

Borrowing

The introduction of investors' borrowing opportunities does not change the necessary conditions for equilibrium given by Eq. (7). Borrowing is simply the "purchase" of a particular type of security. If the j^{th} security is a borrowing contract open to an investor, then its contingent cash "returns" can be written in the same format used above:

$$R_j = [R_j(0), \ldots, R_j(s,t), \ldots].$$

The vector R_j is unusual only in that $R_j(0) > 0$ and $R_j(s,t) \leq 0$.

Selling Short

Selling short can be most conveniently analyzed within the present framework by regarding the short sale of security k as the purchase of a dummy security k* with a vector of contingent returns R_k^* derived from R_k. The vector R_k^* will be roughly a mirror image of R_k. If there are no margin requirements, then $R_k^* = -R_k$, in which case selling security k short is algebraically equivalent to purchasing negative amounts of security k, assuming k* is held to time t = T.[20]

It is entirely feasible to incorporate dummy securities such as k* in the investor's portfolio problem wherever short sales make sense. The necessary conditions for the maximum imply a result comparable to Eq. (7) for each dummy security--that is,

$$P_k^* \geq \sum_{s,t} q(s,t)\, R_k^*(s,t) \,. \tag{9}$$

This holds with an equality if $h_k^* > 0$.

For the case in which there are no margin requirements, comparison of Eqs. (7) and (9) leads to an interesting result. As we have observed, for this case $R_k^* = -R_k$, implying that $P_k^* = -P_k$ and that $R_k^*(s,t) = -R_k(s,t)$ for all (s,t). Substituting in Eq. (9),

$$P_k \leq \sum_{s,t} q(s,t)\, R_k(s,t) \,. \tag{10}$$

Suppose that P_k^* is greater than the right hand side of Eq. (9). This signals that the investor will not want to hold the dummy security k*

in his portfolio. If this is so, however, then from Eq. (10) it follows that P_k, the cost of purchasing security k long, is less than the utility associated with purchase of a marginal share of security k. Thus, if the investor's total expected utility is reduced by selling a marginal amount of security k short, he will necessarily be better off by purchasing a marginal amount of k long. Conversely, if the investor's total expected utility is reduced by purchasing a marginal amount of security k, then it will pay him to sell security k short. Therefore, each investor at equilibrium will be willing to hold at least marginal amounts of each security either long or short in his portfolio. Only if this condition is satisfied will Eqs. (7) and (10) be consistent, implying

$$P_k = \sum_{s,t} q(s,t)\, R_k(s,t) \quad . \tag{11}$$

Note that Eq. (11) requires that all investors agree, at the margin, on the equilibrium values of all securities, although not necessarily on the value of any particular contingent return. Because each investor is willing to "take a position" in each security, there can be no such thing as a "clientele effect." That is, investors holding a particular security will not value it more highly than other investors do.

On the other hand, Eq. (7) is consistent with a clientele effect. Any such effect must therefore be ascribed to restricted trading opportunities, not to the existence of uncertainty, or to differences in investors' expectations.

Other Constraints

The frictions and imperfections which exist in actual markets have, for

the most part, been left out of the above analysis. However, those which impose constraints on investors' portfolio choices can be analyzed with relative ease if portfolio choice is viewed as a problem of non-linear programming.

To illustrate, consider an investor who must keep a certain portion of his portfolio in a particular class of securities. In practice, many institutions are so constrained.

Specifically, suppose the investor must invest at least 100b percent of his funds in securities from the set K. Now the objective function must be maximized subject to two constraints:

$$\phi_1 = \sum_{k=0}^{N} h_k P_k - W = 0,$$

$$\phi_2 = \sum_{k \in K} h_k P_k - bW \leq 0. \tag{14}$$

For securities not included in the set K, $\delta \phi_2 / \delta h_k = 0$. Here the constraint $\phi_2 = 0$ is irrelevant, and Eq. (7) holds. For $k \in K$, however, the Kuhn-Tucker conditions are:

$$\frac{\delta \Psi}{\delta h_k} - \lambda(\phi_1) \frac{\delta \phi_1}{\delta h_k} - \lambda(\phi_2) \frac{\delta \phi_2}{\delta h_k} \leq 0 . \tag{15}$$

Computing $\lambda(\phi_1)$ and the partial derivatives, and solving for P_k, we have:

$$P_k = \frac{1}{U'(0) - \lambda(\phi_2)} \sum_{s,t} \pi(s,t)\, U'(s,t)\, R_k(s,t), \tag{16}$$

assuming that k is actually included in the investor's optimal portfolio. Since $\lambda(\phi_2) \geq 0$, Eq. (16) insures that the investor's demand for security

k is forced higher than it would be otherwise if the constraint is binding.

It is also possible to solve Eq. (16) for $\lambda(\emptyset_2)$:

$$\lambda(\emptyset_2) = U'(0) - \sum_{s,t} \pi(s,t)\, U'(s,t)[R_k(s,t)/P_k] \,. \tag{17}$$

In words, $\lambda(\emptyset_2)$ is the expected utility lost (at the margin) by investing one dollar in a security $k \in K$ instead of consuming the dollar.

IV. A SPECIAL CASE

We now return to the main thread of our argument. A necessary condition for equilibrium if short sales are permitted, and if there are no margin requirements or other imperfections, is that Eq. (11) hold for each investor and each security. For the i^{th} investor, then,

$$P_k = \sum_{s,t} q_i(s,t)\, R_k(s,t), \tag{11}$$

$k = 1, 2, \ldots, N$. In other words, if there are N securities, equilibrium requires that N equations of this form hold for each investor. The "unknowns" are the variables $q_i(s,t)$, since security prices and contingent returns are taken as given by investors in a perfect market. The set $\{q_i(s,t)\}$ represents the present values of contingent returns to the i^{th} investor, given by Eq. (8).

In general, there is no requirement that investors agree on the present value of contingent returns. However, consider the special case in which $N \geq M$, where M is the number of future states, and M of the vectors R_k are linearly independent. Then the equations may be solved to yield a unique set of <u>prices</u> $\{q_i(1,1) \ldots q_i(s,t) \ldots\}$. Moreover, since P_k and R_k

are the same for all investors, the set must be identical for all investors. Given the structure of security prices at equilibrium, we can thus infer an entirely objective set of prices $\{q(s,t)\}$, where $q(s,t)$ is the price at $t = 0$ of one dollar to be paid contingent on the occurrence of (s,t). We have, therefore:

$$P = \sum_{s,t} q(s,t)\, R_k(s,t), \tag{20}$$

with $q(s,t) = q_i(s,t)$ for all i and all (s,t).[21]

In reality, of course, the number of securities is likely to be much less than the number of states. Nevertheless, this simplest possible case is important in several respects.

1. It is customarily argued that, since investors will disagree in their subjective evaluations of the size and risk of streams of future returns, their estimates of the value of these streams will also differ. This may well be true in fact, but Eq. (20) establishes that any such disagreement is not a necessary consequence of either (a) the existence of uncertainty or (b) differences in investors' expectations. In fact, Eq. (20) implies that all investors would agree on the value of any conceivable bundle of contingent returns, no matter how bizzare, which could be specified in terms of the catalogue of contingencies $\{(s,t)\}$.
2. Equation (20) is closely related to (and in a sense depends on) the ability of any investor to achieve any desired pattern of contingent returns from his portfolio. To be specific, let the vector X_p represent the desired pattern:

(21) $$X_P = [X_P(1,1), \ldots, X_P(s,t), \ldots] = \frac{1}{\sum_{s,t} R_P(s,t)} R_P.$$

Here $R_P(s,t)$ is the return of the portfolio in (s,t) and R_P is the vector of these returns. The numbers $X_P(s,t)$ represent the pattern of the contingent returns $R_P(s,t)$. Because $1/\sum_{s,t} R_P(s,t)$ adjusts for the scale of the portfolio's returns, $\sum_{s,t} X_P(s,t) = 1$. The pattern of returns on an individual security can be described similarly:

(22) $$X_k = [X_k(1,1), \ldots, X_k(s,t), \ldots] = \frac{1}{\sum_{s,t} R_k(s,t)} R_k.$$

Since, in this special case, there are M securities with linearly independent vectors X_k, and there are no margin requirements for short sales, the vectors span the M-dimensional space defined by the catalogue of M states. The portfolio vector X_P lies in this same space. It follows that any vector X_P can be obtained by a linear combination of the vectors X_k.

To put this another way, we have established that an investor can adjust his portfolio to change a particular contingent return $R_P(s,t)$, while leaving returns in all other contingencies unchanged. In effect, he can buy or sell returns for any contingency. It is as if there were a separate forward market for dollars to be delivered in each future state. Viewed in this light, it is not surprising that a unique set of prices $\{q(s,t)\}$ is a necessary condition for equilibrium.

3. Previous time-state-preference models have, without signifi-

> cant exception, confined their analysis to this special case. In fact, it is usually assumed that trading of contingent returns takes place in explicit markets, rather than implicitly, via portfolio adjustments. Arrow and Hirshliefer, for instance, have assumed markets for "primitive securities": [22] the primitive security for (s,t) pays one dollar contingent on (s,t), but nothing in any other state. Thus the equilibrium price of such a security would be simply q(s,t). [23]

Without denying the theoretical productivity of this special case, [24] it is important to recognize that the time-state-preference framework is amenable to considerable generalization and adaptation to particular market characteristics. I hope the above analysis has established this point.

V. SOME IMPLICATIONS

This section notes some general implications of the time-state-preference model of security valuation. First, the conventional valuation formulas are briefly re-examined. Observations follow on the implications of individual risk aversion for market prices, the interpretation of time-state-preference models if the catalogue of states is not exhaustively defined, and the concept of a risk-equivalent class of securities.

Conventional Valuation Formulas

Consider the i^{th} investor, who holds at least one share of the k^{th} stock. Then Eq. (11) holds at equilibrium:

$$P_k = \sum_{s,t} q_i(s,t)\, R_k(s,t). \tag{11}$$

This investor may or may not agree with others on the present value of

contingent returns. For simplicity's sake, however, we will drop the subscript i in what follows.

The formulas normally used are:

$$(1,2) \qquad P = \sum_{t=1}^{T} \frac{\bar{R}_k(t)}{(1+r)^t} = \sum_{t=1}^{T} \frac{\alpha(t)\ \bar{R}_k(t)}{(1+i)^t} \ .$$

These formulas may be regarded as simplifications of Eq. (11). Thus the required rate r or the coefficients $\alpha(t)$ depend on (1) the pattern across states of stock k's contingent dividends, (2) the investor's valuations of contingent returns, and (3) his probability assessments.

For instance, it is shown in the appendix that

$$(23) \qquad \alpha(t) = \frac{Z_k(t)\ Q(t)'}{Z_k(t)\ \overline{\pi}(t)'} \ ,$$

where:

$$Z_k(t) = \frac{1}{\sum_{s=1}^{m(t)} R_k(s,t)} [R_k(1,t), \ \ldots \ , R_k(m(t),t)],$$

a vector expressing the pattern of contingent returns paid by security k at time t;

$$Q(t) = \frac{1}{\sum_{s-1}^{m(t)} q(s,t)} [q(1,t), \ \ldots \ , q(m(t),t)],$$

a vector expressing the relative "prices" for returns contingent on states at time t;

$$\overline{\pi}(t) = [\overline{\pi}(1,t), \ \ldots \ , \overline{\pi}(m(t),t)],$$

a vector of the investor's probability assessments for states at time t.

Of course, there is no guarantee that investors will agree on the appropriate size of the coefficients $\alpha(t)$, since they need not agree on the vectors Q(t) or $\overline{\pi}(t)$. However, Eq. (23) holds from the point of view of

any investor considered singly.[25]

Equations (1) and (2) are two among many ways of simplifying the more basic valuation formula, Eq. (11). Alternative forms based on continuous compounding and exponential growth are often seen, as are rules of thumb using price-earnings ratios or "multipliers." Given a little algebraic ingenuity, the possible formats are endless.

If one accepts the time-state-preference model as general, it is pointless to say that any particular simplification is the correct way to compute present value. The choice among algebraic formats depends on intuitive appeal, tractability and the appropriateness of the assumptions necessary to the various simplifications. The intelligent user will choose the method that best fits the case at hand.

Risk Aversion

The next few paragraphs investigate the implications of investors' risk aversion for the structure of security prices. The conclusions are generally consistent with those obtained elsewhere,[26] but they bear repetition because of the persistence of the notion that security prices are adequately explained by simply considering the characteristics of individual investors' utility functions.

It is generally accepted that most investors are risk averse. From this, it is often inferred that "the market" should be risk averse, in the sense that the certainty equivalent of an uncertain return should always be no more than the expectation of the return. In other words, the prediction would be that $\alpha(t) \leq 1$, or that $r \geq i$, where i is the pure rate of interest.

Actually, it is always possible to construct patterns of contingent

returns for which $\alpha(t) > 1$ for all t, --i.e., such that r > i. Note that the numerator and denominator of Eq. (23) are weighted averages of relative prices and probabilities, respectively. In general, the relative price for (s,t) may be more or less than $\pi(s,t)$. By changing the weights $Z_k(t)$, therefore, we can always assure that $Z_k(t)Q(t)' > Z_k(t)\pi(t)'$, or that $\alpha(t) > 1$.[27] The economic meaning of this manipulation is that an expected return will be more valuable if it pays relatively higher returns in states in which contingent returns have a relatively high value.

On the other hand, suppose the weights $Z_k(t)$ are chosen randomly, subject to the condition that the elements of $Z_k(t)$ sum to one. The expected result of this experiment is that $\alpha(t) = 1$. (Note that this result does not depend on the law of large numbers. If $\alpha(t) \neq 1$, it is due to a sampling error, not to a lack of opportunities for diversification.) Securities constructed in this manner would, on the average, be priced to yield exactly the pure rate of interest.

These mental experiments indicate that a risk premium is not explained by uncertainty per se, but by some systematic relationship between the relative sizes of the returns $R_k(s,t)$ and the "prices" $q(s,t)$. In actual markets, the relationship seems to be that returns on most available securities are positively correlated, so that securities tend to pay high returns precisely when most portfolio returns are high and low returns in times of scarcity. The normal risk premium is thus explained, given the inverse relationship between supplies of contingent returns and their present values.

One prediction which may be drawn from this argument is that investors will require no risk premium to gamble, providing that:

a. The gambling is on a small enough scale that the vector Q_i can be taken as given.

b. The outcome of the gamble is not related to the state from the set $\{(s,t)\}$ which actually occurs. That is, the gamble must be an "artificially generated risk," not a "natural hazard."[28]

Interpretation of the Model Given Coarse Partitions of States[29]

The application of the time-state-preference model within a relatively coarse partition of the set S of possible event-sequences is entirely feasible, given attention to several complicating factors. One of these is that investors will not, in general, adopt identical partitions, so that agreement among investors on the risk characteristics of securities cannot be taken for granted. Nevertheless, postulating agreement will often prove to be appropriate.

Other problems arise regarding the definition of a contingent return. Given the partition $\{(\sigma,t)\}$ which is coarser than $\{(s,t)\}$, the returns contingent on (σ,t) are the random variables $\tilde{R}_k(\sigma,t)$. They cannot be used in the same sense as the variables $R_k(s,t)$--which are certain returns, given (s,t)--without further explanation.

Adopting the partition $\{(\sigma,t)\}$, the investor's decision problem is to maximize

$$\psi = \sum_{\sigma,t} \pi(\sigma,t)E[U(\sigma,t)] + U(0), \tag{24}$$

subject to a wealth constraint, where

$$E[U(\sigma,t)] = E[U(\sum_{k=1}^{N} h_k\tilde{R}_k(\sigma,t))]. \tag{25}$$

The value of $E[U(\sigma,t)]$ could be computed readily if the investor had specified the returns $R_k(s,t)$, the probabilities $\pi(s,t)$ (given (σ,t)), and the functions $U(s,t)$; but he does not have this information. A reasonable heuristic tool is to rewrite his decision problem as:

(24a) $$\Psi = \sum_{\sigma,t} \pi(\sigma,t)\, U^*(\sigma,t) + U(0),$$

(25a) $$U(\sigma,t) = U\left(\sum_{k=1}^{N} h_k CEQ_k(\sigma,t)\right).$$

Here $CEQ_k(\sigma,t)$ is the certainty equivalent of $\tilde{R}_k(\sigma,t)$--that is, if state (σ,t) occurs, the investor is indifferent to receiving $\tilde{R}_k(\sigma,t)$ or a certain/$\underset{\text{amount}}{CEQ_k(\sigma,t)}$. The investor is assumed to act as if he is certain to receive a portfolio return of $\sum_{k=1}^{N} h_k CEQ_k(\sigma,t)$ if (σ,t) occurs.

This is an easy way out only if the certainty equivalent can itself be explained without undue complication. Various simple relationships might be assumed if the partition represented by the set $\{(\sigma,t)\}$ is not too coarse. If, for instance, the partition is fine enough to describe all systematic interrelationships among security returns, then one could postulate that $CEQ_k(\sigma,t) = E[\tilde{R}_k(\sigma,t)]$. The reasoning here is as follows.[30] If there are no systematic interrelationships among security returns, given (σ,t), then the returns $\tilde{R}_k(\sigma,t)$ of the N securities are independent random variables. Under these conditions, one would expect to find that the random return $\tilde{R}_k(\sigma,t)$ and a certain return of the amount $E[\tilde{R}_k(\sigma,t)]$ are equally valuable, given that (σ,t) occurs. That is, this would be the expected result if the investor assessed the present value of $\tilde{R}_k(\sigma,t)$ with the set of states $\{(s,t)\}$ in mind.[31] There is, of course, some

chance for error: the coarser the partition, the less likely it is that the postulated relationship will be an adequate approximation.

"Risk Classes" as a Consequence of Coarse Partitions

In a time-state-preference framework, the risk characteristics of the k^{th} security are determined by its pattern of returns across the possible states of nature. The vector X_k, defined by Eq. (22), is one way to describe this pattern.

Unfortunately, it is not very helpful to say that securities j and k are in the same risk class if $X_j = X_k$, for this requires the returns $R_k(s,t)$ and $R_j(s,t)$ to be perfectly correlated in every contingency. If this is true, there is little point in calling j and k different securities. The use of such a definition would thus require creating a risk class for every security, and it implies that no two securities can be considered perfect substitutes.

This is not surprising, considering that individuals are assumed to have made the computational investment necessary to evaluate securities within the set of states $\{(s,t)\}$. A smaller computational effort yields a coarser partition, and a corresponding reduction in the investor's ability to distinguish among the risk characteristics of securities. If computation is costly, it is perfectly conceivable that an investor will consider the j^{th} and k^{th} securities to be perfect substitutes, knowing that $X_j \neq X_k$, but not being able to specify the differences among the two (because of a coarse partition of future states) in any which would allow a choice among them.[32] Thus the concept of a class of securities with homogeneous risk characteristics--found useful by Modigliani and Miller,

for instance[33]--is not unreasonable if computational effort is a scarce resource.

VI. THE INTERDEPENDENCE OF INVESTORS' STRATEGIES

The model of security valuation presented in this paper is not realistic in any strict sense. No theory is. On the other hand, the assumptions used are mostly familiar ones; few readers will be surprised to encounter such abstractions as the Perfect Market or the Rational Investor.

The one novel assumption is that all investors purchase portfolios at $t = 0$ with the certain intention of holding them unchanged until a (distant) horizon at $t = T$. This proviso insures that investors' portfolio choices are independent, in the sense that the expected utility of any investor's portfolio depends only on the cash returns of the securities included, and in no way on possible future actions of other investors.

It takes only cursory observation of actual security markets to see that this assumption of a "one-shot" portfolio choice is inaccurate. Investors' strategies are clearly interdependent, for instance, if securities are purchased partly for anticipated capital gains. Here the return realized by any particular investor depends not only on the state (s, t) occurring, but also on what other investors think the security is worth.

The interdependence of investors' strategies is a matter of considerable theoretical interest and uncertain practical importance. This section is a brief, qualitative examination of the problems involved.

Why Investors Revise Their Portfolios

There are two sorts of reasons why an investor may sell securities

from his original portfolio before the horizon period at $t = T$.

1. To provide funds for consumption.--An investor may sell securities if the cash returns on his portfolio do not sustain his desired consumption expenditures. Some of these consumption needs, such as retirement income, are fairly predictable, but others are not: security investment serves in part as a cushion or reserve source of funds which may be needed unexpectedly for other uses.

It is important, however, to look one step behind this proximate cause of the sale of securities. Our previous assumption of a one-shot investment decision is not necessarily inconsistent with an investor's providing exactly for a large contingent cash payment, since there will be some portfolio with a pattern of returns across the set of states which is appropriate. If this pattern lies within the cone spanned by the vectors X_k of available securities, then the investor can purchase a portfolio now to meet these contingent needs precisely.[34] For instance, a certain requirement for x dollars of retirement income t years hence can be provided for in advance by purchasing government bonds of the appropriate maturity.

However, such opportunities do not generally exist for all types of consumption needs, since the actual number of securities is too small to span more than a small portion of the different patterns of portfolio returns which may be desired. Moreover, the problem is only partially solved by postulating the "special case" in which an investor can obtain any conceivable pattern in the vector space defined by the set of states $\{(s,t)\}$. Suppose, for instance, that an investor perceives the possibility of a personal emergency at $t = 1$. He will not be able to provide

for the emergency situation by his portfolio choice unless securities exist which give different returns contingent on the occurrence of the emergency. Unless it is related in some way to economic conditions on a broader scale, this will not be the case. One would not expect to find securities offering different returns contingent on the occurrence of an event of purely personal interest.[35] Even in this special case, therefore, an investor's need for a large amount of money income contingent on a personal event cannot always be met without portfolio adjustments when the event occurs.

2. Portfolio choice is a sequential decision problem. Whereas the contingent needs just discussed are needs for funds to be consumed,[36] investors may also wish to reinvest these funds in other securities. There are two motives for this sort of trading. First, the passage of time and the resolution of uncertainty yield changes in the risk characteristics and relative prices of available securities as well as the investor's probability judgments and preferences. His attitudes to risk may also change. The result is that the portfolio which is optimal at $t = 0$ may not be at $t = 1$. Second, the investor may believe he can anticipate changes in security prices accurately enough so that speculative trading is advisable.

This second class of reasons for investors to anticipate the possible future sale of securities is particularly important, since these considerations can be formally analyzed only by explicit treatment of portfolio choice as a sequential decision problem. The nature of this problem may be indicated by noting that, in our model, the marginal utility to an investor of money in (s,t) is dependent only on his portfolio choice at

$t = 0$, since the returns yielded by the portfolio are determined solely by this choice. In general, however, the return received in (s,t) also depends on (a) the opportunities which develop before time t and (b) the investor's strategy in pursuing these opportunities. In this more general case, the marginal utility of income in (s,t) cannot be deduced solely from consideration of the initial portfolio choice. The result is that this variable cannot be derived and used to evaluate contingent returns in (s,t) without further analysis.

Unfortunately, none of the conventional models of security valuation allow adequate formal analysis of this sequential, stochastic decision problem. The framework presented in this paper is similarly ill-equipped, at least at the present stage of its development.

Treatment of These Problems in the Literature of Finance

The problems raised by the interdependence of investors' strategies have been recognized, but not emphasized, in the literature. In essence, what has been done is to assume that these problems have no systematic effect on the valuation process.

Suppose we begin by comparing (a) an investor's valuation of an incremental share of a security on the assumption that he will hold the share until the ultimate horizon, time T, to (b) his valuation of this share assuming that it is to be sold in some period $t < T$. The bundle of contingent returns he receives in case (b) differs from (a) in the substitution of the security's price at t for the contingent dividends paid by the firm between t and T. Since the level and risk characteristics of the security's price at time t are closely associated with those of the security's bundle of contingent returns subsequent to that time, it is a reasonable

first approximation to assume that the present value of the price at t and the bundle of subsequent returns is the same. Given this assumption, the value of any security can be expressed solely as a function of its contingent cash returns.

This argument, which has been widely used in the literature,[37] also justifies any of the results which can be obtained by use of the basic time-state-preference model presented above.

The difficulty is that the risk characteristics of a security price at some future date t are also dependent on all investors' demands for this security at that time. These demands cannot be predicted with certainty (given the security's contingent payments up to time t), since the preferences and strategies of investors are not determined solely by the state of nature which obtains. (Remember that we have defined states only in terms of contingent cash returns on securities; the catalogue does not cover all conceivable events.) Therefore, the investor who may sell a security at time t is exposed to uncertainty about other investors' future demands in addition to the uncertainty inherent in its bundle of subsequent contingent returns. This price uncertainty is precisely why the interdependence of investors' strategies is potentially important to any theory of security valuation. Its actual importance cannot be determined here, but the next subsection cites two situations in which it is likely to be relevant.

Situations in Which the Interdependence of Investors' Strategies is Likely to be Relevant

1. Short-term price fluctuations.--The random, seemingly senseless, short-term fluctuations of stock prices, to say nothing of the speculative

bubbles that occur from time to time, are regarded by many as less related to changes in the present value of future cash returns than to other, "irrational" factors. Often this conclusion is reached too casually. First, volatility does not establish irrationality, since there is nothing irrational about changing one's mind. Second, "irrationality" is a loose word. It seems to refer to price changes unrelated to factors relevant in stable equilibria populated by Economic Man.

But if it does exist, this irrationality may be attributed to one or both of two reasons. First, investors may tend to make mistakes in the pursuit of their own interests. Second, investors may be "concerned, not with what an investment is really worth to a man who buys it 'for keeps,' but with what the market will value it at, under the influence of mass psychology, three months or a year hence."[38] In this case, the interdependence of investors' strategies is dominant.[39]

2. Commitments to Future Sale or Purchase of Securities.--Interesting theoretical problems are not always empirically relevant. Suppose we predicted the structure of security prices on the assumption that the prices depend only on the securities' contingent cash returns. Could we then improve our prediction by taking the interdependence of investors' strategies into account? Such an improvement could take place only if (a) securities differ in ways not reflected in their sets of contingent cash returns and (b) these differences are relevant to the investor because of the interdependence of investors' strategies.

Some such differences might be found where the "irrational" behavior just discussed seems most prevalent; but we will not belabor this topic

further. To take another tack, note that the commitment to buy or sell a security at a future date (or in a particular state of nature) clearly exposes the investor to price uncertainty; the extent of exposure depends on the extent of the commitment. We would expect security prices to be affected by the interdependence of investors' strategies where strong commitments are common. It should suffice here to cite two examples.

The many studies of the term structure of interest rates have investigated the effects, if any, of price uncertainty. The liquidity premium found by most such studies is interpreted as an extra payment made by holders of short-term bonds for protection from price uncertainty. It may not be clear, however, how a commitment to buy or sell bonds is involved.

If an investor "needs" a certain amount of funds in ten years, we might refer to t = 10 as his "natural habitat,"[40] since a bond maturing at that time would be ideal for him. Higher anticipated yields on bonds of different maturities may lure him from his natural habitat, but if he does so he is exposed to price uncertainty. If the "need" is in fact given, purchasing a five-year bond now commits him to buy another bond at t = 5, and bond yields at that time are uncertain. On the other hand, buying a fifteen-year bond effectively commits him to selling, at an uncertain price, at t = 10. Thus an investor can be said to commit himself to future sales or purchases when he forsakes his natural habitat. These commitments are one consideration explaining the liquidity premiums just noted.[41]

A second type of implied commitment is found in much corporate borrowing, evidenced by the frequent refinancing of corporate issues. Most firms borrow for relatively short periods, compared with the de facto maturities

of their assets. When this is done, the firm commits itself to refinancing when the borrowed funds are due. If new borrowing is to be undertaken, the firm's shareholders are indirectly exposed to the price uncertainty reflected in uncertainty about the level and term structure of interest rates.

To be sure, the commitment to borrow is not absolute, since the shareholders always have the option of providing additional future financing themselves. This is easily done by retention of earnings if the firm is not highly leveraged;[42] otherwise, new securities would have to be issued. Unfortunately, the effects of price uncertainty are not avoided in either case. If refinancing by shareholders is anticipated, ownership of the firm's shares implies a commitment to make an additional investment in some future period or contingencies. In general, there is no guarantee that such an investment is consistent with portfolios which would otherwise be optimal at that time.[43] If it is not consistent, we would expect an adverse effect on the present price of the firm's shares. The magnitude of the effect would depend on the firm's debt-equity ratio and the disparity between the maturity structures of its assets and liabilities.

VII. CONCLUDING NOTE

Hieshleifer has remarked that "one surprising aspect of the time-and-state preference model is that it leads to a theory of decision under uncertainty while entirely excluding the 'vagueness' we usually associate with uncertainty."[44] It should now be clear that such precision is not a necessary characteristic of all time-state-preference models, but only of the special case Hirshleifer was concerned with. Given a limited number of

securities, restrictions on short sales, and the possible effects of the interdependence of investors' strategies, a certain amount of vagueness seems unavoidable. Thus Eq. (7), the most basic valuation formula, is an inequality.

I do not find this particularly discouraging, since the effects of such imperfections as restrictions on short sales will appear in any model. Vagueness seems to be characteristic of actual problems, not of the models we invent to solve them.

To be sure, it will often be sufficient to assume that prices are determined as if the world were perfectly precise. But we have shown that time-state-preference models are still useful when vagueness is unavoidable.

APPENDIX

The problem is to show that, if $\alpha(t)$ is defined so that

$$(2,11) \qquad P_k = \sum_{s,t} q(s,t)R_k(s,t) = \sum_{t=1}^{T} \frac{\alpha(t)\bar{R}_k(t)}{(1+i)^t} \quad ,$$

then

$$(23) \qquad \alpha(t) = \frac{Z_k(t)Q(t)'}{Z_k(t)\pi(t)'} \quad .$$

Where $Z_k(t)$, $Q(t)$ and $\pi(t)$ are the vectors defined on p. 23.

The value of one dollar delivered in every state at time t is $\sum_{s=1}^{m(t)} q(s,t)$. The riskless interest rate is thus defined by

$$\frac{1}{(1+i)^t} = \sum_{s=1}^{m(t)} q(s,t) \quad .$$

Noting the definitions of the vectors $Q(t)$ and $Z_k(t)$, we can write for Eq. (2):

$$P_k = \sum_{t=1}^{T} \frac{Z_k(t)Q(t)' \sum_{s=1}^{m(t)} R_k(s,t)}{(1+i)^t}$$

For each period, multiply by

$$\frac{\bar{R}_k(t)}{\bar{R}_k(t)} = \frac{\bar{R}_k(t)}{Z_k(t)\,\pi(t)' \sum_{s=1}^{m(t)} R_k(s,t)} = 1 \quad .$$

Equation (2) becomes

$$P_k = \sum_{t=1}^{T} \frac{1}{(1+i)^t} \frac{Z_k(t)\ Q(t)'}{Z_k(t)\ \pi(t)} \bar{R}_k(t) \quad .$$

Thus the coefficient $\alpha(t)$ may be expressed by Eq. (23).

FOOTNOTES

1. This paper is a further development of my doctoral dissertation [24], which was submitted to the Graduate School of Business, Stanford University, in 1967. I am indebted for good advice and apt suggestions to my dissertation committee, Professors Alexander Robichek, Gert von der Linde and Ezra Solomon. Also, Professor Kenneth Arrow was kind enough to read and comment on the entire dissertation. I wish also to thank Professors Jack Hirshleifer, Avraham Beja and Peter Diamond for helpful comments.

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3. The framework is due to Arrow [2] and has been extended and expounded by Debreu [5], esp. ch. vii, and Hirshleifer [10] [11] [12]. See also Radner [27], Drèze [7], Pye [26], Diamond [4] and Beja [3] for examples of related work. Lancaster [16] has used similar analytical framework in recent discussions of theory of consumer choice.

4. In the articles already cited.

5. In the articles already cited.

6. Lack of space constrains me from detailed analysis of particular financing or investment problems. This is done in my dissertation [24]. For an

earlier version of part of the analysis presented there, see Robichek and Myers [29]. For other applications of a time-state-preference model, see Hirshleifer [11], esp. pp. 264-68.

7. Although current, or expected average, earnings are often used as a proxy for the dividend stream.

8. See Robichek and Myers [30].

9. See Robichek and Myers [28], and Lintner [19], 27-28.

10. The classic exposition for the certainty case is Fisher's [8]. Also, see Hirshleifer [13].

11. Markowitz [20], Sharpe [31], and Lintner [18] [19].

12. See Lintner [19], p. 27.

13. The choice of a particular partition is arbitrary. An even coarser partition than that used here would undoubtedly be more "realistic," since investors would in practice regard computational efforts as a scarce resource. The intuitive meaning of a still finer partition is difficult to pin down, if only because no one person is likely to be <u>interested</u> in more than a small subset of the conceivable event sequences comprising the set S.

The interpretation of time-state-preference models given coarser partitions than $\{(s,t)\}$ is discussed in Section V, below.

14. The horizon t = T is introduced primarily for analytical convenience. There is some error because of the lack of explicit analysis of events

subsequent to the horizon, but the effect of any such errors on the market's valuation of securities at t = 0 may be considered negligible if the horizon is far enough distant in time.

15. It is true that the contingent returns received by stock- and bondholders are affected if the firm replaces equity with debt. However, the bundles of contingent returns offered by the firm's securities can still be clearly specified within the set of states $\{(s,t)\}$, provided that (a) there is no change in the total contingent returns paid by the firm on all its outstanding securities and (b) investors are certain about how the firm's total payout is to be divided among stock- and bondholders in every possible contingency. Although these conditions may not always hold in practice (see Robichek and Myers [29], esp. pp. 15-19) they are a reasonable approximation for present purposes.

16. However, there is no requirement that all securities may pay returns in all time periods from t = 1 to t = T. Bonds, in particular, will often mature before the horizon period.

17. As noted by Hirshleifer [12], pp. 523-34.

18. Kuhn and Tucker [15]. Also, see Dorfman, Samuelson and Solow [6], ch. vii, for the exposition which prompted my use of the conditions. Remember that the conditions to be presented are not sufficient for equilibrium. For instance, one necessary condition not mentioned is that the utility functions U(s,t) be convex--i.e., risk-averse. See Arrow [2], p. 95. Also, in the absence of any direct or indirect restraints on the ability of investors to sell single contingent payments, we must require that

$\pi_i(s,t) > 0$ for all investors (indexed by i) and all (s,t). If an investor really believes that the contingency (s,t) is impossible, he will be willing to sell contingent payments in (s,t) in unlimited amounts. This latter point was mentioned to me by Avraham Beja. For a detailed treatment of the existence of possible equilibrium states, see Debreu [5].

19. Since from the nature of the problem the constraint Eq. (4) must be satisfied exactly, $\lambda(\emptyset)$ cannot be zero.

20. Given the distant horizon T, the short sale becomes a promise to pay security k's dividends from period t = 1 to t = T to the lender of the security. The payments include the security's price at t = T, which we have interpreted as a liquidating dividend. Thus selling short is the sale of future contingent returns. That we do not actually find short sales undertaken as long-term commitments is apparently due to uncertainty about whether any particular investor could fulfill such a contract. Margin requirements are a reaction to this uncertainty.

21. This result may hold even if short sales are restricted. But this requires that (a) the vectors of returns of available securities--including borrowing and any "dummy securities" used to describe types of trading different from simple purchases--span a cone equivalent to the M-dimensional space created by the set $\{(s,t)\}$, and (b) that Eq. (7) holds with an equality for all securities and all investors. Note that in this case the number of securities would have to be substantially more than the number of states.

22. See Arrow [2]. Hirshleifer calls these primitive securities "time-

state claims." [12], p. 527, and passim.

23. The set of primitive securities and the M normal securities considered above are simply alternative bases for the vector space defined in terms of $\{(s,t)\}$.

24. For instance, the special case has generated considerable insight into the problem of determining optimal capital structure for corporations. See Robichek and Myers [29] and Hirshleifer [11], pp. 264-68.

25. Confidence in the theoretical appropriateness of certainty equivalents may be somewhat increased by finding that they can be conveniently expressed in a time-state-preference framework. Unfortunately, the required rate r cannot be conveniently expressed--a fact which corroborates Lintner's view that r is not a "primary" variable for theoretical uses. See Lintner [19], pp. 27-28.

26. See, for instance, Drèze [7], pp. 36-38; Lintner [19], pp. 22-23.

27. If $\pi(t) = Q(t)$, $\alpha(t) = 1$ for any pattern on contingent returns. This is improbable.

28. The terms are Hirshleifer's. [12], p. 532.

29. The following comments are not subject to the same standards of precision as is the exposition of the "basic model" in Section III above. Nevertheless, they may be helpful in indicating the options open to model-builders within the time-state-preference framework.

30. Note that a parallel argument concerning the certainty equivalent

coefficient $\alpha(t)$ is given above, p. 25.

31. This procedure presumes the investor to be indifferent to the vagueness created by use of the set of states $\{(\sigma,t)\}$ instead of $\{(s,t)\}$. Whether such indifference represents rational behavior is an interesting open question.

32. However, because of the general benefits of diversification, the investor may hold both securities in his portfolio.

33. See Modigliani and Miller [22]. These comments are not meant to imply that the concept of a risk class is necessary to the proof of Modigliani and Miller's Proposition I--that the market value of the firm is independent of financial leverage in the absence of taxes on corporate income. The proposition can be readily proved given the detailed partition defined by $\{(s,t)\}$, in which no two securities can be said to belong to the same risk class. See Hirshleifer [11], pp. 264-68, and Robichek and Myers [29].

34. Such a portfolio would yield relatively high returns in the states in which the contingent needs occur, but relatively low returns otherwise. This case illustrates the pitfalls of always associating risk aversion with the variability of a portfolio's contingent returns.

35. It may be possible for an investor to issue securities which are differentiated in this regard. We see this in practice as insurance. But many risks are not insurable so that we can count on some emergencies remaining.

The reasons why investors usually cannot issue securities to cover

all contingent needs have been discussed by Radner [27] and Arrow[1], pp. 45-56. Transaction costs are an obvious reason. Another is the difficulty of writing a contract in which the duties of the parties depend on which state of nature actually occurs, when the catalogue of states is not exhaustively defined. A third reason is that the very existence of a contract may change the subsequent actions of the parties to it, in turn affecting the probabilities of occurrence of the states on which the contract is contingent. As Arrow [1] notes (p. 55) this problem arises in practice when insurance policies may make the issuing company vulnerable to a "moral hazard."

36. An important assumption made in our model is that all contingent returns are consumed--i.e., not reinvested in securities.

37. For example, see Gordon [9], pp. 131-32, Porterfield [25], p. 19, Lintner [19], p. 27. Lintner uses a slight variant of this assumption in another paper. See [17], p. 69.

38. Keynes [14], pp. 154-55.

39. Some of the irrational aspects could be ruled out by assuming "symmetric market rationality," which requires that every individual (a) is rational in pursuit of his own interests and (b) imputes rationality to the market. See Miller and Modigliani [21], pp. 427-28. However, this is not sufficient to make investors' strategies independent.

40. The term is Modigliani and Sutch's [23].

41. They are not sufficient explanations. For instance, see Modigliani

and Sutch [23], pp. 183-84.

42. Robichek and Myers [29] discuss how the necessity to refinance may effect the optimal degree of leverage for highly leveraged firms.

43. In the "special case" discussed in Section IV above, however, the investor could always offset the effect of the additional investment by short sales and/or sale of other securities in his portfolio. Thus the commitment to invest additional amounts does not constrain his portfolio choice. In less idealized worlds, the commitment may be binding.

44. Hirshleifer [12], p. 534.

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